

# GLOBAL STRUCTURE OF QUATERNION POLYNOMIAL DIFFERENTIAL EQUATIONS

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**ABSTRACT.** In this paper we mainly study the global structure of the quaternion Bernoulli equations  $\dot{q} = aq + bq^n$  for  $q \in \mathbb{H}$  the quaternion field and also some other form of cubic quaternion differential equations. By using the Liouvillian theorem of integrability and the topological characterization of 2-dimensional torus: orientable compact connected surface of genus one, we prove that the quaternion Bernoulli equations may have invariant tori, which possesses a full Lebesgue measure subset of  $\mathbb{H}$ . Moreover, if  $n = 2$  all the invariant tori are full of periodic orbits; if  $n = 3$  there are infinitely many invariant tori fulfilling periodic orbits and also infinitely many invariant ones fulfilling dense orbits.

## 1. INTRODUCTION AND MAIN RESULTS

The dynamics of ordinary differential equations in  $\mathbb{R}$  or  $\mathbb{C}$  has been intensively studied from many different points of view. While because of the noncommutativity of the quaternion algebra, the study on quaternion differential equations becomes very difficult and much involved, and the results in this field are very few. Recent years because of their application in quantum and fluid mechanics, see e.g. [2, 3, 12, 16, 17, 30, 29], the study on the dynamics of quaternion differential equations has been attracting more interesting.

In 2006 Campos and Mawhin [10] initiated the study on the existence of periodic solutions of one-dimensional first order periodic quaternion differential equations. Wilczyński [31] continued this study and paid more attention on the existence of two periodic solutions of quaternion Riccati equations. Our work in [15] presented a study on the global structure of the quaternion autonomous homogeneous differential equations

$$(1) \quad \dot{q} = aq^n, \quad q \in \mathbb{H},$$

where  $a \in \mathbb{H}$  is a parameter. Recall that  $\mathbb{H}$  is the quaternion field.

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<sup>1</sup>The author is partially supported by NNSF of China grant 10831003 and Shanghai Pujiang Program grant 09PJ013.

2000 *Mathematics Subject Classification.* 34C25, 34C37, 34K05, 37K10.

*Key words and phrases.* Quaternion field, polynomial differential equations, global structure, Liouvillian integrability, torus.

In this paper we will study the global dynamics of the quaternion Bernoulli equations

$$(2) \quad \dot{q} = bq + aq^n,$$

with  $a, b, q \in \mathbb{H}$ ,  $2 \leq n \in \mathbb{N}$  and also of the third order equation

$$(3) \quad \dot{q} = a(q - c_0)(q + c_0)q,$$

with  $a \in \mathbb{H}$  and  $c_0 \in \mathbb{R}$ .

The quaternion Bernoulli equation (2) consists of linear terms and homogenous nonlinearities of degree  $n$ . We note that real planar polynomial vector fields generalizing the linear systems with homogeneous nonlinearities have been extensively studied from different points of view, for instance limit cycles, centers, phase portraits and integrability, see e.g. [18, 14, 23, 25]. Some famous three dimensional real differential systems exhibiting chaotic phenomena, for instance Lorenz system, Rabinovich systems and Rikitake systems and so on, also have this form, which consist of linear terms and homogeneous nonlinearity of degree 2. As our knowledge the dynamics of the quaternion equations of form (2) with  $a, b \neq 0$  has never been studied. We note that for either  $a = 0$  or  $b = 0$ , equation (2) is in fact the equation (1), and it has been studied in [15].

Equation (2) with  $a, b \neq 0$  can be written in

$$(4) \quad \dot{q} = a(cq - q^n),$$

with  $a, c \in \mathbb{H}$  not zero.

Our first result is the following.

**Theorem 1.** *For the quaternion differential equation (4) with  $c \in \mathbb{R}$  not zero, the following statements hold.*

(a) *Assume that  $a + \bar{a} \neq 0$  and  $a - \bar{a} = 0$ .*

(a<sub>1</sub>) *The phase space  $\mathbb{R}^4$ , i.e.,  $\mathbb{H}$ , is foliated by invariant planes of (4), which all pass through the origin.*

(a<sub>2</sub>) *On each invariant plane, there are  $n$  singularities: one is the origin and the others are located on the circle centered at the origin with the radius  $\sqrt[n-1]{|c|}$ , denoted by  $S_c$ . All non-trivial orbits are heteroclinic, and connect the origin and one of the singularities on  $S_c$  except the following  $2(n-1)$  ones: there are exactly  $n-1$  heteroclinic orbits connecting the origin and the infinity, and also  $n-1$  ones connecting each one of the singularities on  $S_c$  and the infinity.*

(b) *Assume that  $a^2 - \bar{a}^2 \neq 0$ .*

(b<sub>1</sub>) *Each orbit of system (4) starting on the branch of  $P := \{q^{n-1} + \bar{q}^{n-1} - c = 0\}$  is heteroclinic connecting the origin and one of the singularities given by  $q^{n-1} = c$ , which are located in two consecutive region limited by the branches of  $P$ .*

- (b<sub>2</sub>) *There exists at least one orbit in each connected region limited by the branches of  $P$ , which connects the infinity and one of the singularities of (4).*
- (c) *Assume that  $a + \bar{a} = 0$  and  $a - \bar{a} \neq 0$ .*
  - (c<sub>1</sub>) *The hypersurfaces  $P$  are invariant, on which all orbits are non-trivial and located in two dimensional invariant algebraic varieties.*
  - (c<sub>2</sub>) *The invariant set  $\mathbb{R}^4 \setminus \{P\}$  is foliated by one invariant plane, two 2-dimensional invariant algebraic varieties and 2-dimensional invariant tori. The invariant plane is foliated by  $n$  isochronous centers with  $n$  separatrices going to infinity. One of the algebraic varieties is full of singularities and the other fulfils periodic orbits with a center and finitely many heteroclinic orbits.*

A *nontrivial orbit* is an orbit which is not a singularity. An *algebraic variety* is a subset of  $\mathbb{R}^4$  formed by the common zeros of finitely many polynomials.

In statement (c<sub>2</sub>) of Theorem 1 we do not study the dynamics of equation (4) on the invariant tori. In fact, the next theorem shows that the dynamics on the invariant tori depend on the degree  $n$  of the equations.

Now we study the dynamics of equation (4) on the invariant tori appearing in statement (c<sub>2</sub>) of the last theorem for  $n = 2, 3$ . For larger  $n$ , we have no methods to tackle it. The difficulty is the parametrization of the invariant tori as we will see in the proof of the following results.

**Theorem 2.** *For the 2-dimensional invariant tori stated in (c<sub>2</sub>) of Theorem 1 the following statements hold.*

- (a)  $n = 2$ . *Each torus is full of periodic orbits.*
- (b)  $n = 3$ . *Among the tori there are infinite many ones fulfilled periodic orbits and also infinite many ones fulfilled dense orbits.*

The above results are on equation (4) with  $c \in \mathbb{R}$ . We now study the equation with  $c \in \mathbb{H} \setminus \mathbb{R}$ . For general  $a \in \mathbb{H}$  and  $2 < n \in \mathbb{N}$ , we have no method to deal with it. The next result is on equation (4) with  $0 \neq a \in \mathbb{R}$  and  $n = 2$ .

**Theorem 3.** *For equations (4) with  $a \in \mathbb{R}$  nonzero,  $n = 2$  and  $c - \bar{c} \neq 0$ , set  $L = c_0 q_0 + c_1 q_1 + c_2 q_2 + c_3 q_3 - (c_0^2 + c_1^2 + c_2^2 + c_3^2)/2$ , the following statements hold.*

- (a) *If  $c + \bar{c} \neq 0$ , all the orbits of system (4) starting on the hyperplane  $L = 0$  are heteroclinic and spirally approach the singularities  $O = (0, 0, 0, 0)$  and  $S = (c_0, c_1, c_2, c_3)$ . There are other two heteroclinic orbits connecting the infinity and either  $S$  or  $O$ .*

- (b) If  $c + \bar{c} = 0$ , the hyperplane  $L = 0$  is invariant. The invariant set  $\mathbb{R}^4 \setminus \{L = 0\}$  is foliated by one invariant plane foliated by two period annuli, one invariant sphere fulfilling periodic orbits, and 2-dimensional invariant tori.

We remark that the case  $c - \bar{c} = 0$  was studied in Theorems 1 and 2.

Finally we study the cubic quaternion differential equation (3). Without loss of generality we assume  $c_0 > 0$ .

**Theorem 4.** *Consider the cubic equation (3) with  $c_0 > 0$  and  $a \in \mathbb{H}$  nonzero. Set  $L = q_0^2 - q_1^2 - q_2^2 - q_3^2 - c_0^2/2$  and denote by  $L^+$  and  $L^-$  the two sheets of the generalized hyperboloid of  $L = 0$  corresponding to  $q_0 \geq c_0/\sqrt{2}$  and  $q_0 \leq -c_0/\sqrt{2}$ , respectively. The following statements hold.*

- (a) *If  $a + \bar{a} \neq 0$ , any orbit starting on  $L^+$  (resp.  $L^-$ ) is heteroclinic connecting the singularities  $O = (0, 0, 0, 0)$  and  $S_+ = (c_0, 0, 0, 0)$  (resp.  $O$  and  $S_- = (-c_0, 0, 0, 0)$ ).*
- (b) *If  $a + \bar{a} = 0$ , the hyperboloid  $L = 0$  is invariant under the flow of (3). The invariant subset  $\mathbb{R}^4 \setminus \{L = 0\}$  is foliated by periodic orbits and 2-dimensional invariant tori. Of the invariant tori, there are infinitely many ones fulfilling periodic orbits and also infinitely many ones fulfilling dense orbits.*

From Theorems 2 and 4 we conjecture that for quaternion polynomial differential equations of degree larger than 2, if the equations have invariant tori, then of which there are infinitely many ones fulfilling periodic orbits and also infinitely many ones fulfilling dense orbits.

We remark that in the proof of the existence of invariant tori, we will use both the Liouvillian theorem of integrability and also the topological characterization of torus. In the case that the mentioned equations have two functionally independent first integrals but they are not Liouvillian integrable, we prove the existence of invariant tori by showing that the connected parts of the intersection of the level sets of the two first integrals are orientable compact smooth surfaces of genus one.

In this paper, as a by product of our results we find some new class of integrable systems. The problem on searching integrable differential equations, including the integrable Hamiltonian systems, has a long history. It can be traced back to Poincaré and Darboux, and even earlier. In recent years Calogero has done a series of researches in this direction, see for instance [6, 7, 8, 21] and the reference therein.

The paper is organized as follows. In the next section we recall some basic facts on quaternion which will be used later on. In Section 3 we will prove our main results. The last section is the appendix presenting the results for linear quaternion equations.

## 2. BASIC PRELIMINARIES

In this section for readers' convenience we recall some basic facts on quaternion algebra (see e.g., [13, 19, 20]), which will be used later on. Quaternions are non-commutative extension of complex numbers, which are defined as the field

$$\mathbb{H} = \{q = q_0 + q_1i + q_2j + q_3k; q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

with  $i, j, k$  satisfying

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k.$$

For  $a, b \in \mathbb{H}$ , their addition and multiplication are defined respectively as

$$\begin{aligned} a + b &= (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k, \\ ab &= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) + (a_1b_0 + a_0b_1 - a_3b_2 + a_2b_3)i \\ &\quad + (a_2b_0 + a_3b_1 + a_0b_2 - a_1b_3)j + (a_3b_0 - a_2b_1 + a_1b_2 + a_0b_3)k. \end{aligned}$$

Obviously  $a, b \in \mathbb{H}$  commute if and only if the vectors  $(a_1, a_2, a_3)$  and  $(b_1, b_2, b_3)$  are parallel in  $\mathbb{R}^3$ .

For  $a \in \mathbb{H}$ , its conjugate is  $\bar{a} = a_0 - a_1i - a_2j - a_3k$ . Then we have  $\overline{ab} = \bar{b}\bar{a}$ ,  $ab + \bar{b}\bar{a} = ba + \bar{a}\bar{b}$  and  $a\bar{a} = a_0^2 + a_1^2 + a_2^2 + a_3^2$ . The last equality implies that  $(a_1i + a_2j + a_3k)^2 = -(a_1^2 + a_2^2 + a_3^2)$ .

For any  $a \in \mathbb{H}$  nonzero,  $\bar{a}/(a\bar{a})$  is its unique inverse, denoted by  $a^{-1}$ . Moreover, it is easy to check that the elements in  $\mathbb{H}$  satisfy the law of association and distribution under the action of the addition and multiplication.

Mostly we will use the quaternion structures to prove our results. But sometimes it is not enough in the proof, we need to write the quaternion differential equations in components. Considering one-dimensional quaternion ordinary differential equations

$$(5) \quad \dot{q} = \frac{dq}{dt} = f(q, \bar{q}), \quad q \in \mathbb{H},$$

where  $f(q, \bar{q})$  is an  $\mathbb{H}$ -valued function in the variables  $q$  and  $\bar{q}$ . Set

$$f(q, \bar{q}) = f_0(q^*) + f_1(q^*)i + f_2(q^*)j + f_3(q^*)k,$$

where  $q = q_0 + q_1i + q_2j + q_3k$  and  $q^* = (q_0, q_1, q_2, q_3) \in \mathbb{R}^4$ . Then equation (5) can be written in an equivalent way as

$$\dot{q}_s = f_s(q_0, q_1, q_2, q_3) \quad \text{for } s = 0, 1, 2, 3.$$

Last paragraph shows that a one-dimensional quaternion ordinary differential equation is in fact equivalent to a system of four-dimensional real ordinary differential equations. It is well known that the dynamics of higher dimensional real differential systems is usually very difficult to study. Sometimes the existence of suitable invariants is very useful in the study. First integral and invariant algebraic hypersurface are two important invariants. A real valued differentiable function  $H(q, \bar{q})$  is a *first integral* of (5) if the

derivative of  $H$  with respect to the time  $t$  along the solutions of (5) is identically zero. An *invariant algebraic hypersurface* of (5) is defined by the vanishing set of a real polynomial  $F(q, \bar{q})$  satisfying

$$\left. \frac{dF(q, \bar{q})}{dt} \right|_{(5)} = K(q, \bar{q})F(q, \bar{q}),$$

with the *cofactor*  $K(q, \bar{q})$  a real polynomial.

In this paper the most difficult part is the search of invariant algebraic hypersurfaces and of first integrals. Having them we can obtain the dynamics of the equations with the help of qualitative methods. This idea can be found in the study of the Lorenz system [27], of the Rabinovich system [9] and of the Einstein-Yang-Mills Equations [26] and so on.

### 3. PROOF OF THE MAIN RESULTS

**3.1. Proof of Theorem 1.** *Statement (a).* Under the assumption of the theorem we assume without loss of generality that  $a = 1$ , and set  $c = c_0 \in \mathbb{R}$ . Then system (4) can be written in

$$(6) \quad \dot{q} = c_0 q - \frac{q^n + \bar{q}^n}{2} - \frac{q^n - \bar{q}^n}{q - \bar{q}}(q_1 i + q_2 j + q_3 k),$$

for  $q - \bar{q} \neq 0$ , where we have used the fact that  $q - \bar{q} = 2(q_1 i + q_2 j + q_3 k)$  and

$$q^n = \frac{q^n + \bar{q}^n}{2} + \frac{q^n - \bar{q}^n}{q - \bar{q}}(q_1 i + q_2 j + q_3 k).$$

Obviously,  $q^n + \bar{q}^n$  and  $(q^n - \bar{q}^n)/(q - \bar{q})$  are real. Furthermore using the Darboux theory of integrability we can check easily that

$$H_2 = \frac{q_2}{q_1}, \quad H_3 = \frac{q_3}{q_1},$$

are two first integrals of equation (6), which follows from the facts that  $q_1 = 0, q_2 = 0$  and  $q_3 = 0$  are three invariant algebraic hyperplanes with the same cofactor  $c_0 - (q^n - \bar{q}^n)/(q - \bar{q})$ . For more information on the Darboux theory of integrability, see for instance [24, 28].

We remark that the Darboux theory of integrability was developed for polynomial vector fields in  $\mathbb{C}^n$  and  $\mathbb{R}^n$ . Here we can use this theory in the non-commutative field  $\mathbb{H}$ , because the mentioned invariant algebraic hyperplanes and their cofactors are all real. Generally, if a polynomial differential equation in  $\mathbb{H}$  has its invariant algebraic hypersurfaces all real, we can apply the Darboux theory of integrability by using these hypersurfaces.

The existence of the two functionally independent first integrals shows that the  $\mathbb{R}^4$  space is foliated by invariant planes given by  $\{H_2 = h_2\} \cap \{H_3 = h_3\}$  with  $h_2, h_3 \in \mathbb{R} \cup \{\infty\}$ . This proves statement  $(a_1)$ .

We now prove statement  $(a_2)$ , that is, study the dynamics of equation (6) on each invariant plane.

For any  $h_2, h_3 \in \mathbb{R}$ , restricted to each invariant plane  $P_{23} := \{H_2 = h_2\} \cap \{H_3 = h_3\}$  equation (6) becomes

$$(7) \quad \begin{aligned} \dot{q}_0 &= c_0 q_0 - \sum_{s=0}^{[n/2]} \binom{n}{2s} (-\Delta^2)^s q_0^{n-2s}, \\ \dot{q}_1 &= c_0 q_1 - \sum_{s=1}^{[(n+1)/2]} \binom{n}{2s-1} (-\Delta^2)^{s-1} q_0^{n-2s+1}, \end{aligned}$$

where  $[\cdot]$  denotes the integer part function,  $\Delta^2 = q_1^2 + q_2^2 + q_3^2 = q_1^2(1 + h_2^2 + h_3^2)$  and we have used the binormal expansion

$$\begin{aligned} q^n &= \sum_{s=0}^{[n/2]} \binom{n}{2s} (-\Delta^2)^s q_0^{n-2s} \\ &+ \sum_{s=1}^{[(n+1)/2]} \binom{n}{2s-1} (-\Delta^2)^{s-1} q_0^{n-2s+1} (q_1 i + q_2 j + q_3 k), \end{aligned}$$

and the fact that  $(q_1 i + q_2 j + q_3 k)^2 = -\Delta^2$ .

For studying the dynamics of equation (7) we transfer it to the complex field. Set  $z = q_0 + q_1 \sqrt{1 + h_2^2 + h_3^2} i$ . Then equation (7) can be written in a one dimensional complex differential equation

$$(8) \quad \dot{z} = c_0 z - z^n.$$

Clearly, this last equation has  $n$  singularities in  $\mathbb{C}$ :  $z_0 = 0$  and  $z_k = \sqrt[n-1]{|c_0|} \exp\left(i\left(\frac{\delta\pi}{n-1} + \frac{2(k-1)\pi}{n-1}\right)\right)$  for  $k = 1, \dots, n-1$ , where  $\delta = 0$  if  $c_0 > 0$  or  $\delta = 1$  if  $c_0 < 0$ . These singularities are all nodes (see e.g. [4]), and  $z_0 = 0$  has different stability than the other  $n-1$  ones. By introducing the polar coordinates  $z = r e^{i\theta}$  we can prove that equation (8) has exactly  $n-1$  heteroclinic orbits connecting the origin and the infinity, and the unique heteroclinic orbit connecting each  $z_k$  for  $k = 1, \dots, n-1$ , and the infinity. All the other orbits are heteroclinic and connect the origin and one of the  $z'_k$ s. This proves statement (a<sub>2</sub>), and consequently statement (a).

For proving statements (b) and (c), we note that for any  $a \in \mathbb{H}$  there exists a  $c \in \mathbb{H}$  such that  $cac^{-1} = a_0 + \sqrt{a_1^2 + a_2^2 + a_3^2} i$ . Moreover equation (4) with  $c_0 \in \mathbb{R}$  can be transformed to  $\dot{p} = cac^{-1}(c_0 p - p^n)$  by the change of variables  $p = qcq^{-1}$ . So in what follows we assume without loss of generality that

$$a = a_0 + a_1 i.$$

Set

$$H = \frac{(q\bar{q})^{n-1}}{q^{n-1} + \bar{q}^{n-1} - c_0}, \quad S = q^{n-1} + \bar{q}^{n-1} - c_0.$$

We claim that the derivatives of  $H$  and  $S$  along equation (4) are

$$(9) \quad \left. \frac{dH}{dt} \right|_{(4)} = (n-1)(a + \bar{a})(c_0 - H)H,$$

$$(10) \quad \left. \frac{dS}{dt} \right|_{(4)} = (n-1) \left( aq^{n-1}(c_0 - q^{n-1}) + (c_0 - \bar{q}^{n-1})\bar{q}^{n-1}\bar{a} \right).$$

Indeed,

$$\begin{aligned} \left. \frac{d(q^{n-1} + \bar{q}^{n-1})}{dt} \right|_{(4)} &= \sum_{l=0}^{n-2} \left( q^l \dot{q} q^{n-2-l} + \bar{q}^{n-2-l} \dot{\bar{q}} \bar{q}^l \right) \\ &= \sum_{l=0}^{n-2} \left( q^l a c_0 q^{n-1-l} + \bar{q}^{n-1-l} c_0 \bar{a} \bar{q}^l - q^l a q^{2n-2-l} - \bar{q}^{2n-2-l} \bar{a} \bar{q}^l \right) \\ &= (n-1) \left( c_0 (a q^{n-1} + \bar{q}^{n-1} \bar{a}) - (a q^{2n-2} + \bar{q}^{2n-2} \bar{a}) \right). \end{aligned}$$

In the last equality we have used the fact that  $q^l a q^k + \bar{q}^k \bar{a} \bar{q}^l = a q^{k+l} + \bar{q}^{k+l} \bar{a}$ . This proves equality (10). Using equality (10) and the fact that  $q\bar{q}$  and  $q^{n-1} + \bar{q}^{n-1} - c_0$  are real, we can prove easily the equality (9). This proves the claim.

Restricted to the hypersurface  $P := \{S = 0\}$  equation (10) becomes

$$(11) \quad \left. \frac{dS}{dt} \right|_{(4), P} = (n-1)(a + \bar{a})(q\bar{q})^{n-1}.$$

For convenience to the following proof, we write equation (4) in a system.

$$(12) \quad \begin{aligned} \dot{q}_0 &= a_0 \left( c_0 q_0 - \frac{q^n + \bar{q}^n}{2} \right) - a_1 \left( c_0 - \frac{q^n - \bar{q}^n}{q - \bar{q}} \right) q_1, \\ \dot{q}_1 &= a_0 \left( c_0 - \frac{q^n - \bar{q}^n}{q - \bar{q}} \right) q_1 + a_1 \left( c_0 q_0 - \frac{q^n + \bar{q}^n}{2} \right), \\ \dot{q}_2 &= (a_0 q_2 - a_1 q_3) \left( c_0 - \frac{q^n - \bar{q}^n}{q - \bar{q}} \right), \\ \dot{q}_3 &= (a_1 q_2 + a_0 q_3) \left( c_0 - \frac{q^n - \bar{q}^n}{q - \bar{q}} \right) \end{aligned}$$

*Statement (b).* By the assumption we can assume that  $a_0 = 1$ . From (11) we get that if an orbit of (4) passes through  $P$ , it should intersect  $P$  transversally. So each region limited by the branches of  $P$  is either positively or negatively invariant.

Since  $S + c_0$  is a homogeneous polynomial in  $q^* = (q_1, q_2, q_3, q_4) \in \mathbb{R}^4$ , it follows that each branch of  $P$  is either a hyperplane or a generalized hyperboloid. From the expression of  $H$  it follows that each branch of the level hypersurfaces  $H = h$  for  $h \in \mathbb{R}$  (if exist) is compact.

Obviously the level set  $H = 0$  is the origin, and the level set  $H = c_0$  consists of the roots of  $q^{n-1} = c_0$ , because  $H = c_0$  is equivalent to  $(q^{n-1} - c_0)(\bar{q}^{n-1} - c_0) = 0$ . In fact, these level sets are exactly formed by the



singularities. In addition the compact hypersurfaces  $H = h$  approach  $P$  when  $h \rightarrow \pm\infty$ .

The facts from the last paragraph and equation (9) imply that each orbit starting on  $P$  will finally approach two singularities, which are located in two consecutive regions limited by  $P$ . Furthermore, since the function  $H$  has different signs in the two consecutive regions limited by  $P$ , it follows from the continuation of  $H$  in each region limited by  $P$  that any heteroclinic orbit should go to the level set  $H = 0$ , i.e. the origin. This proves statement  $(b_1)$ .

As a by product of the last results we get that for  $0 < h < c_0$  the level set  $H = h$  is empty.

Statement  $(b_2)$  follows from the proof of statement  $(b_1)$ , especially the fact that the orbits starting on two consecutive branches of  $P$  are either all get into or all go out the region limited by the two branches.

*Statement (c).* The assumption means that  $a_0 = 0$  and  $a_1 \neq 0$ . Without loss of generality we take  $a_1 = 1$ .

Set

$$F = q_2^2 + q_3^2.$$

Then  $F$  is a first integral of (4), which follows easily from (12) with  $a_0 = 0$ . Moreover, we get from (9) that  $H$  is also a first integral of (4), which is functionally independent with  $F$ . From (11) it follows that each branch of the hypersurface  $P$  is invariant.

We first study the dynamics of (4) on  $P$ . For  $n = 2$  the level set  $P$  is a hyperplane, on which all orbits are parallel straight lines. For  $n > 2$  some easy calculations show that  $(q^n - \bar{q}^n)/(q - \bar{q}) \neq q^{n-1} + \bar{q}^{n-1}$ . This verifies that system (12) on  $P$  has no singularities. Moreover each orbit on  $P$  is located on a cylinder  $F = f > 0$  and rotates strictly along the cylinder. This proves  $(c_1)$ .

By some direct calculations and using the equality  $\bar{i}q^n + \bar{q}^n i = L_{n-1}(\bar{i}q + \bar{q}i)$ , we get that

$$\begin{aligned} \nabla H &= \frac{2(n-1)(q\bar{q})^{n-2}}{(q^{n-1} + \bar{q}^{n-1} - c_0)^2} \times \\ &\quad \left( \frac{q^n + \bar{q}^n}{2} - c_0 q_0, (L_{n-1} - c_0)q_1, (L_{n-1} - c_0)q_2, (L_{n-1} - c_0)q_3 \right). \end{aligned}$$

So, in the invariant space  $\mathbb{R}^4 \setminus \{P\}$ , the critical points of  $(H, F)$  form the invariant plane  $S_1 := \{q_2 = 0\} \cap \{q_3 = 0\}$ , the invariant varieties  $S_2 := \{(q^n + \bar{q}^n)/2 - c_0 q_0 = 0\} \cap \{L_{n-1} = c_0\}$  and  $S_3 := \{(q^n + \bar{q}^n)/2 - c_0 q_0 = 0\} \cap \{q_1 = 0\}$ , where  $L_{n-1} = (q^n - \bar{q}^n)(q - \bar{q})$ .

We get from (12) that the invariant variety  $S_2$  is full of singularities and that the invariant variety  $S_3$  is full of periodic orbits with a center and the heteroclinic orbits connecting the singularities on  $L_{n-1} = c_0$ .

On the invariant plane  $S_1$ , system (12) is simply

$$\begin{aligned}\dot{q}_0 &= -c_0 q_1 + \sum_{s=1}^{[(n+1)/2]} \binom{n}{2s-1} (-q_1^2)^{s-1} q_0^{n-2s+1}, \\ \dot{q}_1 &= c_0 q_0 - \sum_{s=0}^{[n/2]} \binom{n}{2s} (-q_1^2)^s q_0^{n-2s}.\end{aligned}$$

Taking  $z = q_0 + iq_1$ , the last equation can be written in

$$(13) \quad \dot{z} = i(c_0 z - z^n).$$

Clearly equation (13) has  $n$  singularities: one is at the origin and the others are located on the circle  $|z| = \sqrt[n-1]{|c_0|}$ . Applying Theorem 2.1 of [4] to these singularities we get that the origin is an isochronous center with the period  $2\pi/|c_0|$ , and the other singularities are also isochronous centers with the common period  $2\pi/((n-1)|c_0|)$ . Furthermore, the periodic orbits surrounding the origin have different orientation than the ones around the other singularities. Hence we have obtained the dynamics of equation (13), and consequently that of equation (12) on the critical sets.

For all regular values  $(h, f)$  of  $(H, F)$ , the intersection  $M_{h,f} = \{H = h\} \cap \{F = f\}$  is a two dimensional compact invariant manifold, because  $M_{h,f}$  does not contain singularities and the intersection is transversal. We claim that the connected submanifolds of  $M_{h,f}$  are all invariant tori. Indeed, system (12) can be written in a Hamiltonian system with the Hamiltonian  $H$  under the Poisson bracket  $\{\cdot, \cdot\}$  defined by

$$\{P, Q\} = \nabla P M(q) \nabla Q,$$

where  $P, Q$  are two arbitrary smooth functions in  $R^4$  and

$$M(q) = \frac{(q^{n-1} + \bar{q}^{n-1} - c_0)^2}{2(n-1)(q\bar{q})^{n-2}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Furthermore the first integrals  $H$  and  $F$  are in involution under the Poisson bracket. Then the claim follows from the classic Liouvillian theorem on integrability. For more information on Poisson structures and Liouvillian integrability, see for instance [1, 5]. This proves statement (c).

We complete the proof of the theorem.  $\square$

**3.2. Proof of Theorem 2.** As in the proof of statement (c) of Theorem 1 we take  $a = i$ ,  $c_0 > 0$  and use the notations given there.

*Statement (a).* Equation (12) with  $n = 2$  becomes

$$(14) \quad \begin{aligned}\dot{q}_0 &= (2q_0 - c_0)q_1, & \dot{q}_1 &= c_0 q_0 - q_0^2 + q_1^2 + q_2^2 + q_3^2, \\ \dot{q}_2 &= (2q_0 - c_0)q_3, & \dot{q}_3 &= -(2q_0 - c_0)q_2.\end{aligned}$$

Now

$$P := \{q_0 = c_0/2\}, \quad H = \frac{q_0^2 + q_1^2 + q_2^2 + q_3^2}{2q_0 - c_0}, \quad F = q_2^2 + q_3^2.$$

Recall that  $P$  is an invariant hyperplane and  $H$  and  $F$  are two functionally independent first integrals.

In the invariant space  $\mathbb{R}^4 \setminus \{P\}$ , we have  $2q_0 - c_0 \neq 0$ . For all regular values  $(h, f)$  with  $f > 0$  and either  $h > c_0$  or  $h < 0$ , we will prove that the invariant torus  $M_{h,f} = \{H = h\} \cap \{F = f\}$  is full of periodic orbits. Taking the change of coordinates  $z = q_0 - c_0/2 + i q_1$ ,  $q_2 = r \cos \theta$  and  $q_3 = r \sin \theta$ , equations (14) become

$$(15) \quad \dot{z} = -i z^2 + i \left( r^2 + \frac{c_0^2}{4} \right), \quad \dot{r} = 0, \quad \dot{\theta} = -2 \operatorname{Re}(z),$$

where  $\operatorname{Re}(z)$  denotes the real part of  $z$ . Equations (15) have the solutions

$$\begin{aligned} r(t) &= r, \\ z(t) &= \frac{(z_0 + R + (z_0 - R) \exp(-2Rt i)) R}{z_0 + R - (z_0 - R) \exp(-2Rt i)}, \\ \theta(t) &= \theta_0 + 2 \operatorname{Re} \left( \int_0^t z(s) ds \right) \\ &= \theta_0 + 2Rt + \operatorname{Re} \left( \frac{1}{i} \ln \frac{z_0 + R - (z_0 - R) \exp(-2Rt i)}{2R} \right) \\ &= \theta_0 + 2Rt + \operatorname{Arg} \left( \frac{z_0 + R - (z_0 - R) \exp(-2Rt i)}{2R} \right), \end{aligned}$$

with  $r \in (0, \infty)$  and  $R = \sqrt{r^2 + c_0^2/4}$ . Clearly,  $z(t)$  is a periodic function of period  $\pi/R$  in  $t$ . Moreover, the third part in the summation of the last equality of  $\theta(t)$  is also a periodic function of period  $\pi/R$  in  $t$ . These show that  $q_2$  and  $q_3$  are periodic functions of period  $\pi/R$  in  $t$ , and consequently the orbits on the invariant tori are all periodic. As a by product of the above proof, we get that with the expansion of the tori their periods become smaller and smaller. This proves statement (a).

*Statement (b).* Equation (12) with  $n = 3$  is

$$(16) \quad \begin{aligned} \dot{q}_0 &= -(c_0 - 3q_0^2 + q_1^2 + q_2^2 + q_3^2) q_1, & \dot{q}_1 &= (c_0 - q_0^2 + 3q_1^2 + 3q_2^2 + 3q_3^2) q_0, \\ \dot{q}_2 &= -(c_0 - 3q_0^2 + q_1^2 + q_2^2 + q_3^2) q_3, & \dot{q}_3 &= (c_0 - 3q_0^2 + q_1^2 + q_2^2 + q_3^2) q_2. \end{aligned}$$

Now

$$P := \{2(q_0^2 - q_1^2 - q_2^2 - q_3^2) - c_0\},$$

is an invariant generalized hyperboloid, and

$$H = \frac{(q_0^2 + q_1^2 + q_2^2 + q_3^2)^2}{2(q_0^2 - q_1^2 - q_2^2 - q_3^2) - c_0}, \quad F = q_2^2 + q_3^2,$$

are two functionally independent first integrals.

For each regular values  $(h, f)$  of  $(H, F)$ , we study the dynamics on the invariant tori  $M_{h,f}$ . Since  $f > 0$ , the generalized cylinder  $F = f$  is parametrized by

$$q_2 = \sqrt{f} \cos \theta, \quad q_3 = \sqrt{f} \sin \theta.$$

Restricted to the  $F = f$ , the hypersurface  $H = h$  with  $f^2 + (2f + c_0)h < 0$  can be parametrized by

$$q_0 = \sqrt{G(\varphi, h)} \cos \varphi, \quad q_1 = \sqrt{G(\varphi, h)} \sin \varphi,$$

with

$$G(\varphi, h) = h \cos 2\varphi - f + \sqrt{h^2 \cos^2 2\varphi - 2fh \cos 2\varphi - 2fh - c_0 h}.$$

Note that here we study only those tori  $M_{h,f}$  with  $f > 0$  and  $f^2 + (2f + c_0)h < 0$ . They are probably the most simple ones which can be parametrized.

On the above mentioned invariant torus  $M_{h,f}$ , system (16) writes in

$$(17) \quad \begin{aligned} \dot{h} &= 0, & \dot{\theta} &= c_0 + f - G(\varphi, h)(2 \cos 2\varphi + 1), \\ \dot{r} &= 0, & \dot{\varphi} &= c_0 - G(\varphi, h) \cos 2\varphi + f \cos 2\varphi + 2f. \end{aligned}$$

Set

$$\begin{aligned} A(\varphi, h) &= \sqrt{h^2 \cos^2 2\varphi - 2fh \cos 2\varphi - 2fh - c_0 h}, \\ B(\varphi, h) &= c_0 + f - G(\varphi, h)(2 \cos 2\varphi + 1). \end{aligned}$$

Then

$$c_0 - G(\varphi, h) \cos 2\varphi + f \cos 2\varphi + 2f = \frac{AB}{A + 2h \cos^2 \varphi}.$$

If there is a periodic orbit on  $M_{h,f}$ , we assume that its smallest positive period is  $2m\pi$  in  $\varphi$  and  $2n\pi$  in  $\theta$  for  $m, n \in \mathbb{N}$ . We get from (17) that

$$\int_0^{2m\pi} \left( 1 + \frac{2h \cos^2 \varphi}{A} \right) d\varphi = \int_0^{2n\pi} d\theta.$$

The last equality can be written in

$$2(n - m)\pi = m \int_0^{2\pi} \frac{h(1 + \cos^2 \psi)}{\sqrt{h^2 \cos^2 \psi - 2fh \cos \psi - (2f + c_0)h}} d\psi.$$

We can check easily that

$$I(h) := \frac{1}{2\pi} \int_0^{2\pi} \frac{h(1 + \cos^2 \psi)}{\sqrt{h^2 \cos^2 \psi - 2fh \cos \psi - (2f + c_0)h}} d\psi,$$

is analytic in  $h$  with  $f^2 + (2f + c_0)h < 0$ , and  $I'(h) \not\equiv 0$  in any open subset of  $\mathbb{R}$ . The last claim implies that  $I(h)$  is a locally open mapping. So, for any given  $f > 0$  there exist infinitely many  $h$  such that  $M_{h,f}$  is full of periodic orbits, and also infinitely many  $h$  for which  $M_{h,f}$  has dense orbits. This proves statement (b).

We complete the proof of the theorem.  $\square$

**3.3. Proof of Theorem 3.** Working in a similar way to the proof of Theorem 1, we only need to study equation (4) with  $c = c_0 + ic_1$ . Writing equation (4) in a system gives

$$(18) \quad \begin{aligned} \dot{q}_0 &= c_0 q_0 - c_1 q_1 - q_0^2 + q_1^2 + q_2^2 + q_3^2, & \dot{q}_1 &= c_1 q_0 + (c_0 - 2q_0)q_1, \\ \dot{q}_2 &= (c_0 - 2q_0)q_2 - c_1 q_3, & \dot{q}_3 &= c_1 q_2 + (c_0 - 2q_0)q_3. \end{aligned}$$

Recall that  $L = c_0 q_0 + c_1 q_1 - (c_0^2 + c_1^2)/2$ .

(a) Restricted to the hyperplane  $L = 0$  we have

$$\left. \frac{dL}{dt} \right|_{(18)} = 4c_0(q_0^2 + q_1^2 + q_2^2 + q_3^2).$$

So every orbit intersects the hyperplane  $L = 0$  transversally. Obviously  $L = 0$  is orthogonal to the line connecting the singularities  $O$  and  $S$ , and have the same distance to  $O$  and  $S$ .

Set

$$(19) \quad H = \frac{q_0^2 + q_1^2 + q_2^2 + q_3^2}{2c_0 q_0 + 2c_1 q_1 - K_0},$$

where  $K_0 = c_0^2 + c_1^2$ . We have

$$(20) \quad \left. \frac{dH}{dt} \right|_{(18)} = \frac{-2c_0(q_0^2 + q_1^2 + q_2^2 + q_3^2)B}{(2c_0 q_0 + 2c_1 q_1 - K_0)^2},$$

where  $B = (q_0 - c_0)^2 + (q_1 - c_1)^2 + q_2^2 + q_3^2$ . The level set  $H = h$  is empty if  $0 < h < 1$ , and is a ball, denoted by  $B_h$ , centered at  $(hc_0, hc_1, 0, 0)$  with the radius  $\sqrt{(c_0^2 + c_1^2)(h^2 - h)}$  if  $h > 1$  or  $h < 0$ . We can check easily that the balls  $B_h$  with  $h > 1$  (resp.  $h < 0$ ) contain the singularity  $S$  (resp.  $O$ ) in their interiors and are located in  $L > 0$  (resp.  $L < 0$ ). Furthermore, it is easy to prove that when  $h \searrow 1$  (resp.  $h \nearrow 0$ ) the ball  $B_h$  shrinks to the singularity  $S$  (resp.  $O$ ), and that when  $h \nearrow \infty$  (resp.  $h \searrow -\infty$ ) the ball  $B_h$  expands and approaches the hyperplane  $L = 0$ .

From the derivative of  $H$  and the property of the ball  $B_h$ , it follows that each orbit starting on  $L = 0$  will be heteroclinic connecting the two singularities  $S$  and  $O$ . Moreover we get from the last two equations of (18) that these orbits spirally approach  $S$  and  $O$ . These last proofs imply that except those orbits being heteroclinic to  $S$  and  $O$ , there are two other ones: one is heteroclinic to  $S$  and infinity, and another is heteroclinic to  $O$  and infinity. This proves statement (a).

*Statement (b).* Since  $c_0 = 0$ , we get from (20) that the function  $H$  defined in (19) is a first integral of system (18). Furthermore we can prove that

$$F = \frac{(q_0^2 + q_1^2 + q_2^2 + q_3^2)^2}{(2q_1 - c_1)^2 + 4q_2^2 + 4q_3^2},$$

is also a first integral of system (18). In addition,  $L = 0$ , i.e.  $2q_1 = c_1$  is invariant.

Some calculations show that  $H$  and  $F$  are functionally independent, and that the critical points are  $\{q_2 = 0\} \cap \{q_3 = 0\}$  and  $\{q_0 = 0\} \cap \{-a_1 q_1 + q_1^2 + q_2^2 + q_3^2 = 0\}$ . The corresponding critical values of  $(H, F)$  are  $(h, 0)$  and  $(h, f)$  with  $f > 0$  and  $h = c_1 \sqrt{f}/(2\sqrt{f} - c_1)$ . On the invariant plane  $\{q_2 = 0\} \cap \{q_3 = 0\}$ , there are two period annuli separated by the invariant line  $L = 0$ . On the invariant sphere  $\{q_0 = 0\} \cap \{-a_1 q_1 + q_1^2 + q_2^2 + q_3^2 = 0\}$  all orbits are periodic. This last claim follows from the fact that system (18) restricted to the sphere has the first integral  $q_2^2 + q_3^2$ .

For any  $h > 1$  or  $h < 0$  and  $f > 0$  with  $h \neq c_1 \sqrt{f}/(2\sqrt{f} - c_1)$ , the values  $(h, f)$  are regular for  $(H, F)$ . Since the hypersurfaces  $H = h$  and  $F = f$  are compact and intersect transversally, their intersections denoted by  $M_{h,f}$  should be two dimensional compact invariant manifolds (if exists). We claim that the connected parts of  $M_{h,f}$  are 2-dimensional invariant tori.

We now prove the claim. For doing so, it suffices to show that  $M_{h,f}$  is orientable and has genus 1. Associated to the 2-field

$$\begin{aligned} \nabla H \wedge \nabla F = & \left( \frac{\partial H}{\partial q_0} \frac{\partial F}{\partial q_1} - \frac{\partial H}{\partial q_1} \frac{\partial F}{\partial q_0} \right) \frac{\partial}{\partial q_0} \wedge \frac{\partial}{\partial q_1} + \left( \frac{\partial H}{\partial q_0} \frac{\partial F}{\partial q_2} - \frac{\partial H}{\partial q_2} \frac{\partial F}{\partial q_0} \right) \frac{\partial}{\partial q_0} \wedge \frac{\partial}{\partial q_2} \\ & + \left( \frac{\partial H}{\partial q_0} \frac{\partial F}{\partial q_3} - \frac{\partial H}{\partial q_3} \frac{\partial F}{\partial q_0} \right) \frac{\partial}{\partial q_0} \wedge \frac{\partial}{\partial q_3} + \left( \frac{\partial H}{\partial q_1} \frac{\partial F}{\partial q_2} - \frac{\partial H}{\partial q_2} \frac{\partial F}{\partial q_1} \right) \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial q_2} \\ & + \left( \frac{\partial H}{\partial q_1} \frac{\partial F}{\partial q_3} - \frac{\partial H}{\partial q_3} \frac{\partial F}{\partial q_1} \right) \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial q_3} + \left( \frac{\partial H}{\partial q_2} \frac{\partial F}{\partial q_3} - \frac{\partial H}{\partial q_3} \frac{\partial F}{\partial q_2} \right) \frac{\partial}{\partial q_2} \wedge \frac{\partial}{\partial q_3}, \end{aligned}$$

the dual 2-form is

$$\begin{aligned} \omega = & \left( \frac{\partial H}{\partial q_2} \frac{\partial F}{\partial q_3} - \frac{\partial H}{\partial q_3} \frac{\partial F}{\partial q_2} \right) dq_0 dq_1 - \left( \frac{\partial H}{\partial q_1} \frac{\partial F}{\partial q_3} - \frac{\partial H}{\partial q_3} \frac{\partial F}{\partial q_1} \right) dq_0 dq_2 \\ & + \left( \frac{\partial H}{\partial q_1} \frac{\partial F}{\partial q_2} - \frac{\partial H}{\partial q_2} \frac{\partial F}{\partial q_1} \right) dq_0 dq_3 + \left( \frac{\partial H}{\partial q_0} \frac{\partial F}{\partial q_3} - \frac{\partial H}{\partial q_3} \frac{\partial F}{\partial q_0} \right) dq_1 dq_2 \\ & - \left( \frac{\partial H}{\partial q_0} \frac{\partial F}{\partial q_2} - \frac{\partial H}{\partial q_2} \frac{\partial F}{\partial q_0} \right) dq_1 dq_3 + \left( \frac{\partial H}{\partial q_0} \frac{\partial F}{\partial q_1} - \frac{\partial H}{\partial q_1} \frac{\partial F}{\partial q_0} \right) dq_2 dq_3. \end{aligned}$$

Recall that  $\nabla$  denotes the gradient of a smooth function. Since the fields  $\nabla H$  and  $\nabla F$  are linearly independent on  $M_{h,f}$ , the two form  $\omega$  is non-zero on  $M_{h,f}$ . Hence  $M_{h,f}$  is orientable, see e.g. [1, Sec. 2.5] and also [11].

Denote by  $\mathcal{X}_{h,f}$  the restriction of the vector field defined by (12) to  $M_{h,f}$ . Since the vector field  $\mathcal{X}_{h,f}$  has no singularities, applying the Poincaré–Hopf formula to the manifold  $M_{h,f}$  we get

$$0 = \text{ind}(\mathcal{X}_{h,h_1}) = \chi(M_{h,h_1}) = 2 - 2g,$$

where  $\text{ind}(\mathcal{X}_{h,f})$  denotes the sum of the indices of the singularities of  $\mathcal{X}_{h,f}$  on  $M_{h,f}$ , and  $\chi(M_{h,f})$  and  $g$  are the Euler characteristic and the genus of the surface  $M_{h,f}$ , respectively. This shows that the genus of  $M_{h,f}$  is one. It is well-known that an orientable compact connected surface of genus one is a torus, see e.g., [22, Sec. X] for more details.

We complete the proof of statement (b) and consequently the proof of the theorem.  $\square$

**3.4. Proof of Theorem 4.** Recall that  $L = q_0^2 - q_1^2 - q_2^2 - q_3^2 - c_0^2/2$ . For simplifying the notations we denote by  $H_+$  and  $H_-$  the subset of  $\mathbb{R}^4$  with  $L > 0$  and  $L < 0$  respectively, and by  $H_+^+$  and  $H_+^-$  the two parts of  $H_+$  with  $q_0 > c_0/\sqrt{2}$  and  $q_0 < -c_0/\sqrt{2}$ , respectively.

Working in a similar way to the proof of Theorem 1, we assume without loss of generality that  $a = a_0 + a_1 i$ . Equation (3) is equivalent to the system

$$(21) \quad \begin{aligned} \dot{q}_0 &= a_1 q_1 A - a_0 q_0 B, & \dot{q}_1 &= -a_0 q_1 A - a_1 q_0 B, \\ \dot{q}_2 &= -(a_0 q_2 - a_1 q_3) A, & \dot{q}_3 &= -(a_1 q_2 + a_0 q_3) A, \end{aligned}$$

where  $A = c_0^2 - 3q_0^2 + q_1^2 + q_2^2 + q_3^2$  and  $B = c_0^2 - q_0^2 + 3(q_1^2 + q_2^2 + q_3^2)$ . It is easy to check that system (21) has the three finite singularities  $O$ ,  $S_+$  and  $S_-$ . Furthermore restricted to  $L = 0$  the derivative of  $L$  along the solutions of system (21) with respect to the time  $t$  is

$$(22) \quad \left. \frac{dL}{dt} \right|_{(21)} = -2a_0(2q_0^2 - c_0^2/2)^2.$$

*Statement (a).* We consider the case  $a_0 < 0$ . The proof of the case  $a_0 > 0$  follows from the same arguments than that of  $a_0 < 0$ . By (22) we get that if an orbit intersects  $L = 0$ , it should transversally pass through it. Moreover the orbits meeting  $L = 0$  will go from  $H_-$  to  $H_+$  as the time increases.

Set

$$(23) \quad H = (q_0^2 + q_1^2 + q_2^2 + q_3^2)^2 / (q_0^2 - q_1^2 - q_2^2 - q_3^2 - c_0^2/2).$$

We can check that for  $h \in (2c_0^2, \infty)$  the hypersurface  $E_h := \{H = h\}$  has two branches which are located in  $H_+^+$  and  $H_+^-$  respectively, and that for  $h \in (-\infty, 0)$  the hypersurface  $E_h$  has a unique branch which is located in  $H_-$ . Moreover we can check that for  $h \in (2c_0^2, \infty) \cup (-\infty, 0)$  the hypersurface  $E_h$  is compact and contains one of the three singularities in its interior. When  $h \rightarrow \pm\infty$  the hypersurface  $E_h$  approaches the hyperboloid  $L = 0$ .

Now we can verify that

$$(24) \quad \left. \frac{dH}{dt} \right|_{(21)} = 8a_0 N \frac{(q_0^2 + q_1^2 + q_2^2 + q_3^2)^2}{(c_0^2 - 2(q_0^2 - q_1^2 - q_2^2 - q_3^2))^2},$$

where  $N = (c_0^4 - 2c_0^2(q_0^2 - q_1^2 - q_2^2 - q_3^2) + (q_0^2 + q_1^2 + q_2^2 + q_3^2)^2)$ . Since outside  $S_1$  and  $S_2$  we have  $N > 0$ , it follows that the subsets  $H_+$  (resp.  $H_-$ ) are positively (resp. negatively) invariant by the flow of the system. Furthermore, all orbits starting in  $H_+^+$  (resp.  $H_+^-$ ) will approach  $S_+$  (resp.  $S_-$ ) when  $t \rightarrow \infty$ . All orbits starting in  $H_-$  will go to  $O$  when  $t \rightarrow -\infty$ . So all orbits starting on  $L = 0$  with  $q_0 > c_0/\sqrt{2}$  (resp.  $q_0 < -c_0/\sqrt{2}$ ) will be heteroclinic to  $O$  and  $S_+$  (resp. to  $O$  and  $S_-$ ). This proves statement (a).

*Statement (b).* Equations (22) and (24) show that the hyperboloid  $L = 0$  is invariant and that  $H$  is a first integral of system (21). Moreover we can prove that

$$F = q_2^2 + q_3^2,$$

is also a first integral of (21), and that  $H$  and  $F$  are functionally independent. Recall that  $F = f$  is a 3-dimensional cylinder when  $f > 0$  and is a plane when  $f = 0$ .

Working in a similar way to the proof of statement (b) of Theorem 3 we can prove that for  $h \in (-\infty, 0) \cup (2c_0^2, \infty)$  the intersections  $E_h \cap \{F = f\}$  are either formed by periodic orbits for  $(h, f)$  being critical values or 2-dimensional invariant tori for  $(h, f)$  being regular values. Using the same methods as those given in the proof of statement (b) of Theorem 2 we can prove that of the invariant tori there are infinitely many ones fulfilling periodic orbits and also infinitely many ones fulfilling dense orbits. This proves statement (b) and consequently the theorem.  $\square$

#### 4. APPENDIX: THE LINEAR CASE

For the homogeneous linear differential equations

$$(25) \quad \dot{q} = aq + qb,$$

with  $a, b \in \mathbb{H}$  nonzero, taking  $H = q\bar{q}$  we have

$$\left. \frac{dH}{dt} \right|_{(25)} = (a + \bar{a} + b + \bar{b})(q\bar{q}).$$

Moreover its equivalent 4-dimensional linear differential system has at the origin the four eigenvalues

$$(a + \bar{a} + b + \bar{b})/2 \pm \left( \sqrt{(a - \bar{a})^2} \pm \sqrt{(b - \bar{b})^2} \right) / 2.$$

So the dynamics of (25) follows easily from these eigenvalues.

For the homogeneous linear differential equations

$$(26) \quad \dot{q} = aq + \bar{q}b,$$

with  $a, b \in \mathbb{H}$  nonzero, its equivalent 4-dimensional linear differential system has the four eigenvalues

$$(a - b + \overline{a - b})/2 \pm \sqrt{(a + b - \overline{a + b})^2} / 2, \quad (a + \bar{a})/2 \pm \sqrt{(a - \bar{a})^2 - b\bar{b}} / 2.$$

Then the dynamics of (26) follows easily from these eigenvalues.

For the homogeneous linear equations

$$(27) \quad \dot{q} = aq + b\bar{q},$$

with  $a, b \in \mathbb{H}$  nonzero, its equivalent 4-dimensional linear differential system has the four eigenvalues

$$(a - b + \overline{a - b})/2 \pm \sqrt{(a - b - \overline{a - b})^2} / 2, \quad (a + \bar{a})/2 \pm \sqrt{(a - \bar{a})^2 - b\bar{b}} / 2.$$

Then its dynamics follows also from these eigenvalues.

For the non-homogeneous linear quaternion differential equations

$$(28) \quad \dot{q} = b + aq, \quad \dot{\bar{q}} = \bar{b} + q\bar{a},$$



with  $a, b \in \mathbb{H}$  nonzero, they can be transformed to homogeneous ones via the change of variables  $p = q + a^{-1}b$  or  $p = q + ba^{-1}$ . So their dynamics can be obtained from Theorem 2 of [15].

**Acknowledgements.** The author thanks Professors Armengol Gasull and Jaume Llibre for their discussion and comments to part of results given in the first version of this paper. I should appreciate the referees for their excellent comments and suggestions, which can improve our paper both in mathematics and in the expressions.

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